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Bifurcation Structure of Positive Stationary Solutions for a Lotka-Volterra Competition Model with Diffusion I: Numerical Verification of Local Structure

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1. Introduction

This paper is concerned with the bifurcation structure of positive solutions for the stationary problem

$$(1.1) \qquad \begin{cases} \mathbf{0} = \varepsilon D \mathbf{u}'' + \mathbf{f}(\mathbf{u}), & x \in (0, \pi), \\ \mathbf{u}' = \mathbf{0}, & x = 0, \pi \end{cases}$$

of a Lotka-Volterra competition-diffusion system, where $D = \mathrm{diag}(d_u, d_v)$,

$$\mathbf{u} = (u, v), \quad \mathbf{f}(\mathbf{u}) = (f, g)(\mathbf{u}), \quad f(\mathbf{u}) = f^0(\mathbf{u})\, u, \quad g(\mathbf{u}) = g^0(\mathbf{u})\, v,$$

every parameter is positive, and we call $\mathbf{u}(x) = (u, v)(x)$ *positive* when $u(x)$ and $v(x)$ are positive for any $x \in [0, \pi]$. From the competitive interaction, we may assume that $\mathbf{f}^0(\mathbf{u}) = (f^0, g^0)(\mathbf{u})$ is a smooth function in $\mathbf{u}$ and satisfies

$$f_v^0(\mathbf{u}) < 0, \quad g_u^0(\mathbf{u}) < 0 \quad \text{for any } \mathbf{u} \in \mathbb{R}_+^2,$$

where $\mathbb{R}_+ = (0, +\infty)$. As $\mathbf{f}^0(\mathbf{u})$ is represented as

$$\begin{aligned} f^0(\mathbf{u}) &= f_{0,0}^0 + f_{n_1,0}^0\, u^{n_1} + f_{0,n_2}^0\, v^{n_2} + \text{ the remainder term,} \\ g^0(\mathbf{u}) &= g_{0,0}^0 + g_{n_3,0}^0\, u^{n_3} + g_{0,n_4}^0\, v^{n_4} + \text{ the remainder term} \end{aligned}$$

with suitable constants $f_{i,j}^0$, $g_{i,j}^0$ and n_j, we treat the simplest nonlinearity

$$f^0(\mathbf{u}) = 1 - u^n - c\, v^n, \quad g^0(\mathbf{u}) = 1 - b\, u^n - v^n$$

to discuss the bifurcation structure of positive solutions for (1.1), where n, b and c are positive constants. At this point, it is obvious that (1.1) has constant solutions $(0,0)$, $(0,1)$, $(1,0)$, and

$$\hat{\mathbf{u}}\,(= (\hat{u}, \hat{v})) = \left(\left(\frac{1-c}{1-bc} \right)^{\frac{1}{n}}, \left(\frac{1-b}{1-bc} \right)^{\frac{1}{n}} \right)$$

which is positive for either $\max(b, c) < 1$ or $\min(b, c) > 1$.

First of all, let us consider the case $\min(b, c) < 1$. Suppose that (1.1) has a positive solution $(u, v)(x)$. Setting

$$\begin{aligned} u(x_-^u) &= \min_{x \in [0,\pi]} u(x), \quad u(x_+^u) = \max_{x \in [0,\pi]} u(x), \\ v(x_-^v) &= \min_{x \in [0,\pi]} v(x), \quad v(x_+^v) = \max_{x \in [0,\pi]} v(x), \end{aligned}$$

we have

$$\begin{aligned} 1 - u(x_-^u)^n - c\, v(x_+^v)^n \le 0 \le 1 - b\, u(x_-^u)^n - v(x_+^v)^n, \\ 1 - b\, u(x_+^u)^n - v(x_-^v)^n \le 0 \le 1 - u(x_+^u)^n - c\, v(x_-^v)^n \end{aligned}$$

by virtue of the functional form of $\mathbf{f}^0(\mathbf{u})$. By the above inequalities, we obtain

$$0 < (1-c)\, v(x_+^v)^n \le (1-b)\, u(x_-^u)^n \le 0 \quad \text{for the case } c < 1 \le b,$$
$$0 < (1-b)\, u(x_+^u)^n \le (1-c)\, v(x_-^v)^n \le 0 \quad \text{for the case } b < 1 \le c.$$

This contradiction implies that (1.1) has no positive solutions for either $c < 1 \le b$ or $b < 1 \le c$. From

$$u(x_+^u)^n - u(x_-^u)^n \le c\, \left(v(x_+^v)^n - v(x_-^v)^n\right) \le b\, c\, \left(u(x_+^u)^n - u(x_-^u)^n\right),$$

we find out that $u(x)$ and $v(x)$ must be constant in x for the case $\max(b,c) < 1$. Hence we see that (1.1) has no positive nonconstant solutions for the case $\min(b,c) < 1$.

Next, let us consider the case

$$\mu = (b,c) \in \mathcal{M} \equiv \{\, (b,c) \mid \min(b,c) > 1 \,\}.$$

It is easy to check that $\mathbf{u} = \hat{\mathbf{u}}$ is an unstable equilibrium point of the ODE $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$, and that for each $n \in \mathbb{R}_+$ and $\mathbf{d} = (d_u, d_v) \in \mathcal{D}(n,\mu)$, the linearized operator of (1.1) around $\mathbf{u} = \hat{\mathbf{u}}$ has the only eigenvalue (respectively, at least two eigenvalues) with positive real part for any $\varepsilon > 1$ (respectively, $0 < \varepsilon < 1$), where

$$\mathcal{D}(n,\mu) = \{\, \mathbf{d} \in \mathbb{R}_+^2 \mid \det(-D + \mathbf{f}_{\mathbf{u}}(\hat{\mathbf{u}})) = 0 \,\}.$$

In brief, $\mathcal{D}(n,\mu)$ consists of $\mathbf{d} \in \mathbb{R}_+^2$ such that the linearized operator with $\varepsilon = 1$ has the eigenvalue 0 whose eigenfunction is of the form $\pm\mathbf{v}\cos x$, where $\mathbf{v}$ is a nontrivial solution of $(-D + \mathbf{f}_{\mathbf{u}}(\hat{\mathbf{u}}))\,\mathbf{v} = \mathbf{0}$. The bifurcation theory gives us the fact that positive nonconstant solutions of (1.1) which look like $\pm\mathbf{v}\cos(k\,x)$ perturbations from $\mathbf{u} = \hat{\mathbf{u}}$ bifurcate at $\varepsilon = 1/k^2$ for any fixed $n \in \mathbb{R}_+$, $\mathbf{d} \in \mathcal{D}(n,\mu)$ and $k \in \mathbb{N}$. As the multiple existence of positive nonconstant solutions for (1.1) is suggested, one important problem is to seek all positive solutions of (1.1). In this paper, as a first step to answer the problem, we shall establish the local bifurcation structure of positive solutions of (1.1) with respect to ε for suitably fixed $\rho = (n, \mu, \mathbf{d})$.

We define the relation $\prec$ by

$$(u_1, v_1) \prec (u_2, v_2) \iff u_1 < u_2,\ v_1 > v_2,$$

and set

$$\mathcal{N} = \bigcup_{(n,\mu)\in\mathbb{R}_+\times\mathcal{M}} \{\,(n,\mu)\,\} \times \mathcal{D}(n,\mu), \quad \mathcal{N}_2 = \{\, \rho \in \mathcal{N} \mid n \ge 2 \,\},$$
$$E_0(\rho) = \mathbb{R}_+ \times \{\,\hat{\mathbf{u}}\,\}, \quad X = \left\{\, \mathbf{u}(.) \in C^2([0,\pi]) \;\middle|\; \mathbf{u}'(0) = \mathbf{0} = \mathbf{u}'(\pi) \,\right\}.$$

For each $\rho \in \mathcal{N}$, we denote by $E(\rho)$ the set of $(\varepsilon, \mathbf{u}(.)) \in \mathbb{R}_+ \times X$ such that $\mathbf{u}(x)$ is a positive solution of (1.1) for ε, and by $E_k(\rho)$ ($k \in \mathbb{N}$) the set of $(\varepsilon, \mathbf{u}(.)) \in E(\rho)$ such that there exists $\ell \in \{0,1\}$ such that $(-1)^{j+\ell}\mathbf{u}'(x) \succ \mathbf{0}$ holds on $(\pi j/k, \pi(j+1)/k)$ for any integer $0 \le j < k$. By definition, we have $\bigcup_{k\ge 0} E_k(\rho) \subset E(\rho)$ for any $\rho \in \mathcal{N}$, and see that $(\varepsilon, \mathbf{u}(.)) \in E_k(\rho)$ is equivalent to $(k^2\varepsilon, \mathbf{u}(./k)) \in E_1(\rho)$ for any $\rho \in \mathcal{N}$ and $k \in \mathbb{N}$.

LEMMA 1.1 ([1]). *$E(\rho) = \bigcup_{k\ge 0} E_k(\rho)$ holds for any $\rho \in \mathcal{N}$.*

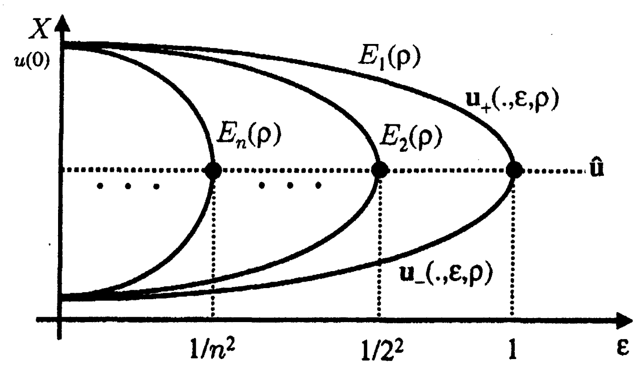


FIGURE 1. Global Bifurcation Structure.

The above lemma says that for each $\rho \in \mathcal{N}$, we can understand the complete structure of $E(\rho)$ by using the information on the structure of $E_1(\rho)$. In consideration of results in [**1**] and [**2**], we may have the following conjecture:

CONJECTURE 1.2. *For any $\rho \in \mathcal{N}$, there exist continuous functions $\mathbf{u}_-(.,\varepsilon,\rho)$ and $\mathbf{u}_+(.,\varepsilon,\rho)$ such that*

(i) $E_1(\rho) = \{\, (\varepsilon, \mathbf{u}_\pm(.,\varepsilon,\rho)) \mid \varepsilon \in (0,1)\,\}$,
(ii) $\pm\mathbf{u}'_\pm(x,\varepsilon,\rho) \prec \mathbf{0}$ *for any* $(x,\varepsilon) \in (0,\pi) \times (0,1)$, *and*
(iii) $\lim_{\varepsilon\to 1} \mathbf{u}_\pm(.,\varepsilon,\rho) = \hat{\mathbf{u}}$.

Figure 1 shows the bifurcation structure of positive solutions for (1.1) which is suggested by the above conjecture. In this paper, to get at the truth of the above conjecture, we shall establish the following result by employing the numerical verification:

THEOREM 1.3. *For each $\rho \in \mathcal{N}_2$, there exist a constant $\nu_0(\rho) > 0$ and C^2-class functions $\varepsilon_0(\nu,\rho)$, $\mathbf{u}_0(.,\nu,\rho)$ defined on the interval $(-\nu_0(\rho), \nu_0(\rho))$ such that*

(i) $(\varepsilon_0(\nu,\rho), \mathbf{u}_0(.,\nu,\rho)) \in E_1(\rho)$ *holds for each* ν, *and*
(ii) $\varepsilon_0(0,\rho) = 1$, $\frac{\partial}{\partial\nu}\varepsilon_0(0,\rho) = 0$ *and* $\frac{\partial^2}{\partial\nu^2}\varepsilon_0(0,\rho) < 0$ *are satisfied.*

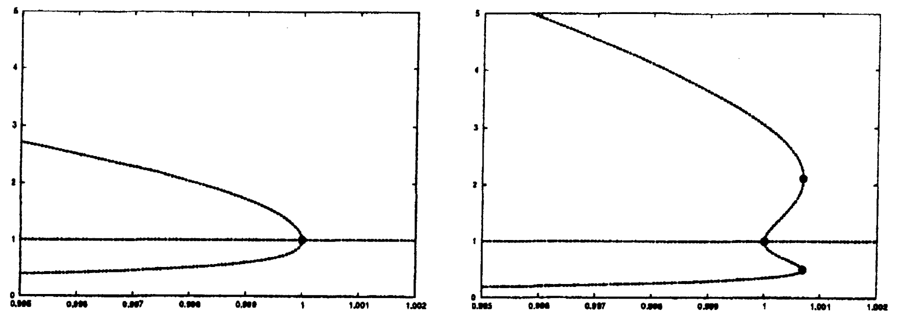
Figure 2 shows numerical bifurcation diagrams for the case where $n = 1.1$ and $d_u = d_v$ are satisfied. The horizontal and vertical axes mean the value of ε and $u(0)/\hat{u}$, respectively. From this figure, we find out that there exists a subregion of $\mathcal{N}$ such that Conjecture 1.2 is not valid.

In this paper, to determine the geometrical position on the curve of positive nonconstant solutions for (1.1) bifurcating from $\mathbf{u} = \hat{\mathbf{u}}$ at $\varepsilon = 1$, we employ the numerical verification method such as the interval arithmetic built into *Mathematica*. Unfortunately, when we change $\mathbf{f}^0(\mathbf{u})$ for

$$f^0(\mathbf{u}) = 1 - u^{n_1} - cv^{n_2}, \quad g^0(\mathbf{u}) = 1 - bu^{n_3} - v^{n_4}$$

with positive constants b, c and n_j, we have not succeeded in establishing the geometrical position, so that the bifurcation structure for (1.1) with more general nonlinearity is still open.

In the next section, we shall discuss the numerical method to verify the fact of Theorem 1.3 only, because we employ Theorem 1.3 and the argument



(a) $b = c = 200.0$ (b) $b = c = 2000.0$

FIGURE 2. Numerical Bifurcation Diagram

in [**2**] and then we can prove that Conjecture 1.2 is valid for each $\rho \in \mathcal{N}_2$ (see [**3**]).

2. Numerical Verification

Let $\rho \in \mathcal{N}_2$ be arbitrarily fixed. Setting

$$\hat{w} = \hat{u}^n, \quad \hat{z} = \hat{v}^n, \quad \hat{\mathbf{w}} = (\hat{w}, \hat{z}), \quad \omega = 1 - \hat{w} - \hat{z}, \quad y = \frac{\hat{z}\, d_u}{n\,\omega},$$
$$\mathcal{J} = (0,1) \times \hat{\mathcal{J}}, \quad \hat{\mathcal{J}} = \left\{ (w,z) \in \mathbb{R}^2_+ \;\middle|\; w + z < 1 \right\},$$

we have $0 < \hat{w} < 1$, $0 < \hat{z} < 1$, $0 < \omega < 1$,

$$\lim_{b \to +\infty,\ c \to +\infty} \hat{\mathbf{w}} = \mathbf{0}, \quad d_v = \frac{n\,(n\,\omega - \hat{z}\, d_u)}{d_u + n\,\hat{w}}, \quad 0 < d_u < \frac{n\,\omega}{\hat{z}},$$

and then we obtain $(y, \hat{\mathbf{w}}) \in \mathcal{J}$ for any $\rho \in \mathcal{N}_2$. We should remark here that although $\mathcal{N}_2$ is an unbounded domain in $\mathbb{R}^5$, $\mathcal{J}$ is a bounded domain in $\mathbb{R}^3$.

We can represent (1.1) as

$$\text{(2.1)} \qquad \begin{cases} 0 = \varepsilon\, d_u\, U'' + \{1 - \hat{w}\, U^n - (1 - \hat{w})\, V^n\}\, U, \\ 0 = \varepsilon\, d_v\, V'' + \{1 - (1 - \hat{z})\, U^n - \hat{z}\, V^n\}\, V, \quad x \in (0, \pi), \\ U' = 0, \quad V' = 0, \quad x = 0, \pi \end{cases}$$

by the change of variables $u = \hat{u}\, U$ and $v = \hat{v}\, V$, and check that the linearized operator of (2.1) around $(U, V) = (1, 1)$ for $\varepsilon = 1$ has the simple eigenvalue 0 with the corresponding eigenfunction $(n\,(1 - \hat{w}), -d_u - n\,\hat{w})\, \cos x$. Substituting

$$\varepsilon = \varepsilon_0(\nu, \rho) = 1 + \nu^2\, \frac{\tilde{\varepsilon}_2(\nu, \rho)}{\sqrt{d_u^2 + d_v^2}},$$
$$U = U_0(x, \nu, \rho) = 1 + \nu\, n\,(1 - \hat{w})\, \cos(\pi\, x) + \nu^2\, \tilde{U}_2(x, \nu, \rho),$$
$$V = V_0(x, \nu, \rho) = 1 - \nu\,(d_u + n\,\hat{w})\, \cos(\pi\, x) + \nu^2\, \tilde{V}_2(x, \nu, \rho)$$

into (2.1), we have

$$\tilde{\varepsilon}_2(0, \rho) = \frac{n^3\, \omega\, r^{(1)}(n, y, \hat{\mathbf{w}})}{r_1^{(2)}(y, \hat{\mathbf{w}})\, r_2^{(2)}(y, \hat{\mathbf{w}})},$$

where

$$r^{(1)}(n, y, \hat{\mathbf{w}}) = 6\, r_0^{(1)}(y, \hat{\mathbf{w}}) + 3\, r_1^{(1)}(y, \hat{\mathbf{w}})\,(n - 2) + r_2^{(1)}(y, \hat{\mathbf{w}})\,(n - 2)^2,$$

and the functions $r_j^{(1)}$ and $r_j^{(2)}$ are shown in Appendix. After simple calculations, we obtain $r_1^{(2)}(y, \hat{\mathbf{w}}) > 0$ and $r_2^{(2)}(y, \hat{\mathbf{w}}) > 0$ for any $(y, \hat{\mathbf{w}}) \in \mathcal{J}$. Hence it follows that the denominator of $\tilde{\varepsilon}_2(0, \rho)$ is positive for any $\rho \in \mathcal{N}_2$.

Hereafter, we shall discuss the numerical method to verify $r_k^{(1)}(y, \hat{\mathbf{w}}) < 0$ for any $(y, \hat{\mathbf{w}}) \in \mathcal{J}$ and $k \in K \equiv \{0, 1, 2\}$. Without loss of generality, we may assume $\hat{w} \le \hat{z}$ by changing the role between u and v if necessary. From

$$\begin{aligned}
r_0^{(1)}(0, \hat{\mathbf{w}}) &= -3\,\hat{w}^4\,\hat{z}^4, & r_0^{(1)}(1, \hat{\mathbf{w}}) &= -3\,(1-\hat{w})^4\,\hat{z}^2\,(1-\hat{z})^2, \\
r_1^{(1)}(0, \hat{\mathbf{w}}) &= -3\,\hat{w}^4\,\hat{z}^4, & r_1^{(1)}(1, \hat{\mathbf{w}}) &= -3\,(1-\hat{w})^4\,\hat{z}^2\,(1-\hat{z})^2, \\
r_2^{(1)}(0, \hat{\mathbf{w}}) &= -\hat{w}^4\,\hat{z}^4, & r_2^{(1)}(1, \hat{\mathbf{w}}) &= -(1-\hat{w})^4\,\hat{z}^2\,(1-\hat{z})^2,
\end{aligned}$$

we obtain $r_k^{(1)}(y, \hat{\mathbf{w}}) < 0$ for any y in a neighborhood of $y = 0$ and $y = 1$ for each $\hat{\mathbf{w}} \in \hat{\mathcal{J}}$ and $k \in K$. Let $k \in K$ be arbitrarily fixed.

First of all, let us consider the case where $\hat{\mathbf{w}}$ is close to the origin. By

$$\begin{aligned}
r_0^{(1)}(y, \hat{\mathbf{w}}) &= -\,4\,y^4\,(y-1)\,(y-2) + o(1), \\
r_1^{(1)}(y, \hat{\mathbf{w}}) &= -\,y^4\,(y-1)\,(4\,y-10) + o(1), \\
r_2^{(1)}(y, \hat{\mathbf{w}}) &= -\,y^4\,(y-1)\,(4\,y-7) + o(1)
\end{aligned}$$

as $\hat{\mathbf{w}} \to \mathbf{0}$, we should remark here that $r_k^{(1)}(y, \hat{\mathbf{w}})$ is degenerate at $(y, \hat{\mathbf{w}}) = (0, \mathbf{0})$. Since $r_k^{(1)}(y, \hat{\mathbf{w}}) = \sum_{j=0}^{6} r_{k,j}^{(1)}(\hat{\mathbf{w}})\, y^j$ satisfies

$$\begin{aligned}
r_{k,0}^{(1)}(\hat{\mathbf{w}}) &= \tilde{r}_{k,0}^{(1)}\,\hat{w}^4\,\hat{z}^4\,(1+o(1)), & \tilde{r}_{0,0}^{(1)} &= -3, & \tilde{r}_{1,0}^{(1)} &= -3, & \tilde{r}_{2,0}^{(1)} &= -1, \\
r_{k,1}^{(1)}(\hat{\mathbf{w}}) &= \tilde{r}_{k,1}^{(1)}\,\hat{w}\,\hat{z}^3\,(1+o(1)), & \tilde{r}_{0,1}^{(1)} &= -4, & \tilde{r}_{1,1}^{(1)} &= -6, & \tilde{r}_{2,1}^{(1)} &= -3
\end{aligned}$$

as $\hat{\mathbf{w}} \to \mathbf{0}$, it follows that

$$p_k^{(1)}(y, \hat{\mathbf{w}}) \equiv -\frac{r_{k,0}^{(1)}(\hat{\mathbf{w}}) + r_{k,1}^{(1)}(\hat{\mathbf{w}})\,y}{y^2}$$

is positive and strictly decreasing in $y \in \mathbb{R}_+$ for each $\hat{\mathbf{w}} \in \hat{\mathcal{J}}_{k,1}^-$, where

$$\hat{\mathcal{J}}_{k,1}^- = \left\{\, \hat{\mathbf{w}} \in \hat{\mathcal{J}} \;\middle|\; \max\left(r_{k,0}^{(1)}(\hat{\mathbf{w}}), r_{k,1}^{(1)}(\hat{\mathbf{w}})\right) < 0 \,\right\}.$$

Setting $p_k^{(2)}(y, \hat{\mathbf{w}}) = \sum_{\ell=0}^{4} r_{k,\ell+2}^{(1)}(\hat{\mathbf{w}})\, y^\ell$, we have $p_k^{(2)}(1, \hat{\mathbf{w}}) < p_k^{(1)}(1, \hat{\mathbf{w}})$ for any $\hat{\mathbf{w}} \in \hat{\mathcal{J}}$ because of

$$r_k^{(1)}(y, \hat{\mathbf{w}}) = y^2\,\left(p_k^{(2)}(y, \hat{\mathbf{w}}) - p_k^{(1)}(y, \hat{\mathbf{w}})\right).$$

As $\hat{\mathbf{w}} \to \mathbf{0}$, we obtain

$$\begin{aligned}
\tfrac{\partial}{\partial y} p_k^{(2)}(1, \hat{\mathbf{w}}) &= 2\,\tilde{r}_{k,4}^{(1)} + 3\,\tilde{r}_{k,5}^{(1)} + 4\,\tilde{r}_{k,6}^{(1)} + o(1) > 0, \\
\frac{p_k^{(2)}(\hat{z}, \hat{\mathbf{w}})}{\hat{z}^2} &= \tilde{r}_{k,2}^{(1)} + \tilde{r}_{k,3}^{(1)} + \tilde{r}_{k,4}^{(1)} + o(1) < 0, \\
\frac{\tfrac{\partial}{\partial y} p_k^{(2)}(\hat{z}, \hat{\mathbf{w}})}{\hat{z}} &= \tilde{r}_{k,3}^{(1)} + 2\,\tilde{r}_{k,4}^{(1)} + o(1) < 0
\end{aligned}$$

because of

$$
\begin{aligned}
&r_{k,2}^{(1)}(\hat{\mathbf{w}}) = \tilde{r}_{k,2}^{(1)}\,\hat{z}^2\,(1+o(1)), && \tilde{r}_{0,2}^{(1)} = -8, && \tilde{r}_{1,2}^{(1)} = -10, && \tilde{r}_{2,2}^{(1)} = -7,\\
&r_{k,3}^{(1)}(\hat{\mathbf{w}}) = \tilde{r}_{k,3}^{(1)}\,\hat{z}\,(1+o(1)), && \tilde{r}_{0,3}^{(1)} = 14, && \tilde{r}_{1,3}^{(1)} = 10, && \tilde{r}_{2,3}^{(1)} = -4,\\
&r_{k,4}^{(1)}(\hat{\mathbf{w}}) = \tilde{r}_{k,4}^{(1)} + o(1), && \tilde{r}_{0,4}^{(1)} = -8, && \tilde{r}_{1,4}^{(1)} = -10, && \tilde{r}_{2,4}^{(1)} = -7,\\
&r_{k,5}^{(1)}(\hat{\mathbf{w}}) = \tilde{r}_{k,5}^{(1)} + o(1), && \tilde{r}_{0,5}^{(1)} = 12, && \tilde{r}_{1,5}^{(1)} = 14, && \tilde{r}_{2,5}^{(1)} = 11,\\
&r_{k,6}^{(1)}(\hat{\mathbf{w}}) = \tilde{r}_{k,6}^{(1)} + o(1), && \tilde{r}_{0,6}^{(1)} = -4, && \tilde{r}_{1,6}^{(1)} = -4, && \tilde{r}_{2,6}^{(1)} = -4.
\end{aligned}
$$

Since $p_k^{(1)}(y,\hat{\mathbf{w}})$ is positive and decreasing in $y \in \mathbb{R}_+$ for each $\hat{\mathbf{w}} \in \hat{\mathcal{J}}_{k,1}^-$, we have $p_k^{(2)}(y,\hat{\mathbf{w}}) < p_k^{(1)}(y,\hat{\mathbf{w}})$ for any $\hat{\mathbf{w}} \in \hat{\mathcal{J}}_{k,2}^-$ and $y \in [\hat{z},1]$, where

$$
\begin{aligned}
\hat{\mathcal{J}}_{k,2}^- &= \left\{ \hat{\mathbf{w}} \in \hat{\mathcal{J}}_{k,1}^- \;\middle|\; q(\hat{\mathbf{w}}) < 0 \right\},\\
q(\hat{\mathbf{w}}) &= \max\left(r_{k,6}^{(1)}(\hat{\mathbf{w}}), p_k^{(2)}(\hat{z},\hat{\mathbf{w}}), \tfrac{\partial}{\partial y} p_k^{(2)}(\hat{z},\hat{\mathbf{w}}), -\tfrac{\partial}{\partial y} p_k^{(2)}(1,\hat{\mathbf{w}}) \right).
\end{aligned}
$$

By

$$
\begin{aligned}
p_k^{(3)}(y,\hat{\mathbf{w}}) \equiv\,& r_{k,4}^{(1)}(\hat{\mathbf{w}}) + r_{k,5}^{(1)}(\hat{\mathbf{w}})\,y + r_{k,6}^{(1)}(\hat{\mathbf{w}})\,y^2\\
&= \tilde{r}_{k,4}^{(1)} + o(1) < 0,\\
p_k^{(4)}(y,\hat{\mathbf{w}}) \equiv\,& \frac{1}{\hat{z}^2}\left(r_{k,3}^{(1)}(\hat{\mathbf{w}})^2 - 4\,r_{k,2}^{(1)}(\hat{\mathbf{w}})\,p_k^{(3)}(y,\hat{\mathbf{w}}) \right)\\
&= \left(\tilde{r}_{k,3}^{(1)}\right)^2 - 4\,\tilde{r}_{k,2}^{(1)}\,\tilde{r}_{k,4}^{(1)} + o(1) < 0
\end{aligned}
$$

on $[0,\hat{z}]$ as $\hat{\mathbf{w}} \to \mathbf{0}$, we have $p_k^{(2)}(y,\hat{\mathbf{w}}) < 0$ for any $\hat{\mathbf{w}} \in \hat{\mathcal{J}}_{k,3}^-$ and $y \in (0,\hat{z}]$, where

$$
\hat{\mathcal{J}}_{k,3}^- = \left\{ \hat{\mathbf{w}} \in \hat{\mathcal{J}}_{k,2}^- \;\middle|\; \max_{y\in[0,\hat{z}]} \left(p_k^{(3)}(y,\hat{\mathbf{w}}), p_k^{(4)}(y,\hat{\mathbf{w}}) \right) < 0 \right\}.
$$

Hence we obtain $r_k^{(1)}(y,\hat{\mathbf{w}}) < 0$ for any $y \in (0,1)$ and $\hat{\mathbf{w}} \in \hat{\mathcal{J}}_{k,3}^-$. Actually, when we take $\hat{z}_- = 36/600$, we can verify $\{\, \hat{\mathbf{w}} \in \hat{\mathcal{J}} \mid \hat{w} \le \hat{z} \le \hat{z}_- \,\} \subset \hat{\mathcal{J}}_{k,3}^-$ by using the interval arithmetic built into *Mathematica*.

Next, let us consider the case where $\hat{\mathbf{w}} \in \hat{\mathcal{J}}$ and $\hat{z} \ge \hat{z}_+ \equiv 35/600$ are satisfied. By

$$
\{\, \hat{\mathbf{w}} \in \hat{\mathcal{J}} \mid \hat{z} \ge \hat{z}_+ \,\} \subset \{\, \hat{\mathbf{w}} \mid \hat{w} = q\,(1-\hat{z}),\ q \in (0,1),\ \hat{z} \in [\hat{z}_+, 1) \,\},
$$

we may show

$$
r_k^+(y,\hat{z},q) \equiv \frac{r_k^{(1)}(y, q\,(1-\hat{z}), \hat{z})}{(1-\hat{z})^2} < 0
$$

for any $(y,\hat{z},q) \in \mathcal{J}_+ \equiv (0,1) \times [\hat{z}_+,1) \times (0,1)$. To do this, we divide $\mathcal{J}_+$ into rectangular regions such that the length of sides for each region is less than 4^{-7}, examine the sign of $\hat{r}_k^+(y,\hat{z},q)$ for each region by using the interval arithmetic built into *Mathematica*, and then we can verify $\hat{r}_k^+(y,\hat{z},q) < 0$ for any $(y,\hat{z},q) \in \mathcal{J}_+$.

From the above numerical verification, we arrive at $\tilde{\varepsilon}_2(0,\rho) < 0$ for any $\rho \in \mathcal{N}_2$. □

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Appendix A

A.1. Functions.

$$
\begin{aligned}
r_0^{(1)}(y,\hat{\mathbf{w}}) =& -4(1-\hat{w}-\hat{z}-2\hat{w}\hat{z})\omega^3y^6 \;+\; (12-24\hat{w}+12\hat{w}^2-8\hat{z}-43\hat{w}\hat{z}\\
&+51\hat{w}^2\hat{z}-4\hat{z}^2+67\hat{w}\hat{z}^2)\omega^2y^5 \;-\; (8-24\hat{w}+24\hat{w}^2-8\hat{w}^3\\
&+6\hat{z}-104\hat{w}\hat{z}+190\hat{w}^2\hat{z}-92\hat{w}^3\hat{z}-44\hat{z}^2+258\hat{w}\hat{z}^2-179\hat{w}^2\hat{z}^2\\
&-35\hat{w}^3\hat{z}^2+30\hat{z}^3-130\hat{w}\hat{z}^3-35\hat{w}^2\hat{z}^3)\omega y^4 \;+\; \hat{z}(14-77\hat{w}\\
&+112\hat{w}^2-49\hat{w}^3-31\hat{z}+121\hat{w}\hat{z}+11\hat{w}^2\hat{z}-101\hat{w}^3\hat{z}+17\hat{z}^2\\
&-44\hat{w}\hat{z}^2-123\hat{w}^2\hat{z}^2)\omega y^3 \;-\; \hat{z}^2(8-44\hat{w}+133\hat{w}^2-166\hat{w}^3\\
&+69\hat{w}^4-16\hat{z}+79\hat{w}\hat{z}-229\hat{w}^2\hat{z}+145\hat{w}^3\hat{z}+21\hat{w}^4\hat{z}+8\hat{z}^2\\
&-35\hat{w}\hat{z}^2+96\hat{w}^2\hat{z}^2+21\hat{w}^3\hat{z}^2)y^2 \;-\; \hat{w}\hat{z}^3(4-10\hat{w}+33\hat{w}^2\\
&-27\hat{w}^3-4\hat{z}+10\hat{w}\hat{z}-33\hat{w}^2\hat{z})y \;-\; 3\hat{w}^4\hat{z}^4,\\
r_1^{(1)}(y,\hat{\mathbf{w}}) =& -4(1-\hat{w}-\hat{z}-2\hat{w}\hat{z})\omega^3y^6 \;+\; (14-28\hat{w}+14\hat{w}^2-12\hat{z}-37\hat{w}\hat{z}\\
&+49\hat{w}^2\hat{z}-2\hat{z}^2+65\hat{w}\hat{z}^2)\omega^2y^5 \;-\; (10-30\hat{w}+30\hat{w}^2-10\hat{w}^3\\
&-4\hat{z}-84\hat{w}\hat{z}+180\hat{w}^2\hat{z}-92\hat{w}^3\hat{z}-30\hat{z}^2+240\hat{w}\hat{z}^2-177\hat{w}^2\hat{z}^2\\
&-33\hat{w}^3\hat{z}^2+24\hat{z}^3-126\hat{w}\hat{z}^3-33\hat{w}^2\hat{z}^3)\omega y^4 \;+\; \hat{z}(10-71\hat{w}\\
&+112\hat{w}^2-51\hat{w}^3-21\hat{z}+115\hat{w}\hat{z}+5\hat{w}^2\hat{z}-99\hat{w}^3\hat{z}+11\hat{z}^2\\
&-44\hat{w}\hat{z}^2-117\hat{w}^2\hat{z}^2)\omega y^3 \;-\; \hat{z}^2(10-44\hat{w}+127\hat{w}^2-162\hat{w}^3\\
&+69\hat{w}^4-20\hat{z}+75\hat{w}\hat{z}-217\hat{w}^2\hat{z}+141\hat{w}^3\hat{z}+21\hat{w}^4\hat{z}+10\hat{z}^2\\
&-31\hat{w}\hat{z}^2+90\hat{w}^2\hat{z}^2+21\hat{w}^3\hat{z}^2)y^2 \;-3\hat{w}\hat{z}^3(2-4\hat{w}+11\hat{w}^2-9\hat{w}^3\\
&-2\hat{z}+4\hat{w}\hat{z}-11\hat{w}^2\hat{z})y \;-\; 3\hat{w}^4\hat{z}^4,\\
r_2^{(1)}(y,\hat{\mathbf{w}}) =& -4\omega^4y^6 \;+\; (11-11\hat{w}-11\hat{z}-10\hat{w}\hat{z})\omega^3y^5 \;-\; (7-14\hat{w}\\
&+7\hat{w}^2-18\hat{z}-10\hat{w}\hat{z}+28\hat{w}^2\hat{z}+11\hat{z}^2+24\hat{w}\hat{z}^2+10\hat{w}^2\hat{z}^2)\omega^2y^4\\
&-\; 2\hat{z}(2+5\hat{w}-16\hat{w}^2+9\hat{w}^3-7\hat{z}-7\hat{w}\hat{z}-2\hat{w}^2\hat{z}+16\hat{w}^3\hat{z}+5\hat{z}^2\\
&+2\hat{w}\hat{z}^2+18\hat{w}^2\hat{z}^2)\omega y^3 \;-\hat{z}^2(7-20\hat{w}+42\hat{w}^2-52\hat{w}^3+23\hat{w}^4\\
&-14\hat{z}+31\hat{w}\hat{z}-69\hat{w}^2\hat{z}+45\hat{w}^3\hat{z}+7\hat{w}^4\hat{z}+7\hat{z}^2-11\hat{w}\hat{z}^2\\
&+27\hat{w}^2\hat{z}^2+7\hat{w}^3\hat{z}^2)y^2 \;-\hat{w}\hat{z}^3(3-5\hat{w}+11\hat{w}^2-9\hat{w}^3-3\hat{z}\\
&+5\hat{w}\hat{z}-11\hat{w}^2\hat{z})y \;-\hat{w}^4\hat{z}^4,\\
r_1^{(2)}(y,\hat{\mathbf{w}}) =& 12\hat{z}\{\hat{z}^2(1-\hat{w})(1-\hat{z})\cos\theta+(\omega y+\hat{w}\hat{z})^2\sin\theta\},\quad \theta=\tan^{-1}\left(\frac{d_v}{d_u}\right),\\
r_2^{(2)}(y,\hat{\mathbf{w}}) =& -4\omega y^2+5\omega y+\hat{w}\hat{z}.
\end{aligned}
$$

A.2. Source Code.

```
DeleteFile[FileNames["chk.*.math"]];
msh = 16; nlp = 4; bis = Interval[{0, 4}]; vi = Interval[{0, 1}];
dennum = 600; iv = Interval[{0, 36/dennum}]; ivc = Interval[{35/dennum, 1}];
```

```
FN[s_] := FortranForm[N[s]];
FF[s_] := FortranForm[Factor[s]];
(*   Function: SignCheck   *)
SignCheck[y0_, z0_, q0_] := (
Write[stmp, "(*   ", FN[y0], "   ", FN[z0], "   ", FN[q0], "   *)"];
yl = Min[y0]; yd = (Max[y0] - Min[y0])/msh;
zl = Min[z0]; zd = (Max[z0] - Min[z0])/msh;
ql = Min[q0]; qd = (Max[q0] - Min[q0])/msh;
rhnc1 = ReplaceAll[rhnck, {yi -> yd*bis, zi -> zd*bis, qi -> qd*bis}];
Do[zws = zl + 3*zd*iz; zwe = zws + 4*zd;
   Do[qws = ql + 3*qd*iq; qwe = qws + 4*qd;
      rhnc2 = ReplaceAll[rhnc1, {zp -> zws, qp -> qws}];
      Do[yws = yl + 3*yd*iy; ywe = yws + 4*yd;
         rhnc3 = ReplaceAll[rhnc2, {yp -> yws}]; ch = -1;
         Write[stmp, "(*   ", FN[yws], "   ", FN[zws], "   ", FN[qws], "   ",
                     FN[rhnc3], "   *)"];
         If[ch < 0 && Max[rhnc3[[1]]] < 0, ch = 1];
         If[ch < 0 && yws <= 0,
              If[Max[rhnc3[[3]]] <= 0, ch = 1];
              If[Max[rhnc3[[4]]] <= 0 && Max[rhnc3[[2]]] < 0, ch = 1]];
         If[ch < 0 && yws >= 1, If[Min[rhnc3[[3]]] >= 0, ch = 1]];
         If[ch < 0 && iy*(nlp - iy) > 0,
              If[Max[rhnc3[[3]]]*Min[rhnc3[[3]]] > 0, ch = 1];
              If[Min[rhnc3[[4]]] >= 0, ch = 1]];
         If[ch < 0,
              Write[stmp, "flg = 1; SignCheck[ Interval[{ ", Numerator[yws],
                          "/", Denominator[yws], " , ", Numerator[ywe], "/",
                          Denominator[ywe], " }], Interval[{ ", Numerator[zws],
                          "/", Denominator[zws], " , ", Numerator[zwe], "/",
                          Denominator[zwe], " }], Interval[{ ", Numerator[qws],
                          "/", Denominator[qws], " , ", Numerator[qwe], "/",
                          Denominator[qwe], " }] ];"]],
         {iy, 0, nlp}],
      {iq, 0, nlp}],
   {iz, 0, nlp}] );
(*   Computation of Bifurcation Direction   *)
u = 1 + mu*u11*Cos[Pi*x] + mu^2*(u20 + u22*Cos[2*Pi*x]) + mu^3*u33*Cos[3*Pi*x];
v = 1 + mu*v11*Cos[Pi*x] + mu^2*(v20 + v22*Cos[2*Pi*x]) + mu^3*v33*Cos[3*Pi*x];
du = (du0 + (mu*ep1 + mu^2*ep2)*Cos[th])/(Pi*Pi);
dv = (dv0 + (mu*ep1 + mu^2*ep2)*Sin[th])/(Pi*Pi);
u11 = n*(1 - w);
v11 = - (du0 + n*w);
vs  = - (du0 + n*w)*us/(n*(1 - z));
p1 = Collect[TrigReduce[Normal[Series[du*D[u, {x, 2}]
        + u*(1 - w*u^n - (1 - w)*v^n), {mu, 0, 3}]]], mu];
p2 = Collect[TrigReduce[Normal[Series[dv*D[v, {x, 2}]
        + v*(1 - z*v^n - (1 - z)*u^n), {mu, 0, 3}]]], mu];
p3i = Integrate[TrigReduce[(p1*us + p2*vs)*Cos[Pi*x]], x];
p3 = Factor[ReplaceAll[p3i, x -> 1] - ReplaceAll[p3i, x -> 0]];
p1tbl = Table[Coefficient[p1, mu, k], {k, 1, 3}];
p2tbl = Table[Coefficient[p2, mu, k], {k, 1, 3}];
p3tbl = Table[Coefficient[p3, mu, k], {k, 1, 3}];
ep1 = Factor[ReplaceAll[ep1, First[Solve[p3tbl[[2]] == 0, ep1]]]];
ep2 = Factor[ReplaceAll[ep2, First[Solve[p3tbl[[3]] == 0, ep2]]]];
dv0 = Factor[ReplaceAll[dv0, First[Solve[p2tbl[[1]] == 0, dv0]]]];
u22 = Factor[ReplaceAll[u22, First[Solve[D[p1tbl[[2]], x] == 0, u22]]]];
v22 = Factor[ReplaceAll[v22, First[Solve[D[p2tbl[[2]], x] == 0, v22]]]];
u20 = Factor[ReplaceAll[u20, First[Solve[p1tbl[[2]] == 0, u20]]]];
v20 = Factor[ReplaceAll[v20, First[Solve[p2tbl[[2]] == 0, v20]]]];
ep2 = Factor[ReplaceAll[ep2, du0 -> n*(1 - w - z)*y/z]];
ep2n = Factor[Numerator[ep2]];
ep2d = Factor[Denominator[ep2]];
rh2 = Collect[Cancel[ep2d/(12*z*(z^2*(1 - w)*(1 - z)*Cos[th]
```

```
                        + ((1 - w - z)*y + w*z)^2*Sin[th]))], y];
rh1 = Collect[Cancel[ep2n/(n^3*(1 - w - z))], y, Factor];
If[Exponent[rh1, n] != 2, Quit[0]];
(*   Denominator   *)
sch = Max[Cancel[ReplaceAll[rh2, y -> 0]/(w*z)],
          Cancel[ReplaceAll[rh2, y -> 1]/((1 - w)*(1 - z))],
          - Cancel[D[rh2, {y, 2}]/(1 - w - z)]];
If[!NumericQ[sch] || sch <= 0, Quit[0]];
(*   Numerator   *)
ftr = { w^4*z^4, w*z^3, z^2, z, 1, 1, 1 };
chktbl = Factor[Table[ReplaceAll[D[rh1, {n, k}]/k!, n -> 2],
                      {k, 0, Exponent[rh1, n]}]];
chktbl[[1]] = Cancel[chktbl[[1]]/6]; chktbl[[2]] = Cancel[chktbl[[2]]/3];
(*   *)
Do[ph1 = Normal[Series[chktbl[[k]], {y, 0, 1}]];
   Do[sch = Max[ReplaceAll[Collect[Cancel[Coefficient[ph1, y, l]/ftr[[l + 1]]],
                                   z], {w -> iv, z -> iv}]];
      If[!NumericQ[sch] || sch >= 0, Quit[0]], {l, 0, 1}];
(*   *)
   ph2 = Cancel[(chktbl[[k]] - ph1)/y^2];
   sch = Max[ReplaceAll[Collect[Coefficient[ph2, y, 4], z],
                        {w -> iv, z -> iv}],
             - ReplaceAll[Collect[ReplaceAll[D[ph2, y], y -> 1], z],
                        {w -> iv, z -> iv}],
             ReplaceAll[Collect[Cancel[ReplaceAll[ph2, y -> z]/z^2], z],
                        {w -> iv, z -> iv}],
             ReplaceAll[Collect[Cancel[ReplaceAll[D[ph2, y], y -> z]/z], z],
                        {w -> iv, z -> iv}]];
   If[!NumericQ[sch] || sch >= 0, Quit[0]];
(*   *)
   ph3 = Cancel[ReplaceAll[ph2, y -> yc*z]/z^2];
   sch1 = Coefficient[ph3, yc, 0];
   sch2 = Coefficient[ph3, yc, 1];
   sch3 = Cancel[(ph3 - sch1 - sch2*yc)/yc^2];
   sch4 = Cancel[(sch2^2 - 4*sch1*sch3)/(1 - w - z)];
   sch5 = ReplaceAll[Collect[{ sch3, sch4 }, z], {w -> iv, z -> iv, yc -> wi}];
   sch = Max[sch5[[1]], sch5[[2]]];
   If[!NumericQ[sch] || sch >= 0, Quit[0]],
   {k, 1, Length[chktbl]}];
(*   *)
Do[rhtmp = Factor[ReplaceAll[chktbl[[k]], w -> q*(1 - z)]/(1 - z)^2];
   rhnck = { rhtmp, Cancel[Coefficient[rhtmp, y, 1]/(q*z^3)],
             D[rhtmp, y], D[rhtmp, {y, 2}] };
   rhnck = Collect[ReplaceAll[rhnck, {y -> yp + yi, z -> zp + zi, q -> qp + qi}],
                   {yi, zi, qi, yp, qp, zp}];
   fnm = 1; flg = 1;
   While[flg > 0,
         stmp = OpenWrite["chk.n" <> ToString[k] <> "." <> ToString[fnm] <> ".math",
                          FormatType -> OutputForm, PageWidth -> 400];
         If[fnm == 1,
            SignCheck[wi, ivc, wi],
            flg = -1;
            Get["chk.n" <> ToString[k] <> "." <> ToString[fnm - 1] <> ".math"]];
            Close[stmp]; fnm++],
   {k, 1, Length[chktbl]}];
(*   End of Job   *)
Quit[0];
```